Successive deceleration in Boltzmann-like traffic equations

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When modeling the slowing-down process in kinetic traffic flow equations the assumption of an instantaneous deceleration is usually made. In this paper we consider a successive slowing-down process, where drivers react on traffic conditions ahead of them in a more moderate manner. From this modified interaction we derive macroscopic traffic equations. Although the present model equations seem to be rather similar to a traffic model presented recently [C. Wagner *et al.*, Phys. Rev. E **54**, 5073 (1996)], the changes result in qualitatively different behavior of the velocity variance when a traffic cluster builds up, i.e., the velocity variance decreases, whereas it increases in the preceding ''free driving region.'' This dynamical behavior is supported by empirical data and is complementary to the behavior found in former macroscopic models. [S1063-651X(97)04406-1]

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I. INTRODUCTION

In recent years the theory of vehicular traffic has gained increasing attention [1–10]. One interesting aspect is the link between microscopic and macroscopic descriptions. For microscopic simulations cellular automata are widely used, but already Prigogine and Herman and others [11–14,10] have introduced an approach using Boltzmann-like equations to describe traffic flow. On the macroscopic side several models have been proposed [15,9,3], mainly based on analogies to hydrodynamic equations. In a series of papers Helbing [2] investigated the theoretical foundations of macroscopic traffic equations and derived Euler-like and Navier-Stokes-like models from a reduced version of Paveri-Fontana's equation [12]. In a recent paper [1], we derived Euler-like traffic equations from the full version of Paveri-Fontana's equation, thus showing that a spatial variation in the variance of the desired speed can cause the onset of a traffic jam. Furthermore, we have extended Paveri-Fontana's equation to high densities by taking the finite car length into account similarly to Enskog's theory [16–18] for dense gases, eventually leading to different gradient terms in the macroscopic equations.

We now resume this work and generalize another assumption made when writing down Paveri-Fontana's equation, the assumption of an instantaneous slowing-down process. We relax this assumption towards a successive deceleration by introducing a modified version of the interaction term. In Sec. II we briefly summarize Paveri-Fontana's traffic equation and then introduce our modified interaction term for extended vehicles with successive deceleration. The macroscopic equations are derived in Sec. III and are related to our former model. We find that, besides some marginal changes, the most important different feature is the appearance of a highly nonlinear term in the variance equation of the actual velocity that has an impeding effect. Indeed, the numerical simulations discussed in Sec. IV show that this term yields a qualitatively different behavior of the variance of the actual velocity when compared to former models. In the congested region the variance now decreases, whereas it increases in the preceding ''free driving region.'' This dynamical behavior is a direct consequence of the different interaction term and is supported by empirical data.

II. THE MODIFIED BOLTZMANN-LIKE EQUATION

In a recent paper [1], we started from a Boltzmann-like traffic equation proposed by Paveri-Fontana [12]. In the original work by Paveri-Fontana some assumptions were made when modeling the deceleration process as a collision integral. For example, the vehicles are assumed to be point-like particles and the slowing-down process is instantaneous. The former assumption has already been generalized in [1] by taking into account the finite car length and a velocity-dependent safety distance analogous to the classical approach for dense gases due to Enskog [16–18]. Now we want to relax the assumption of an instantaneous slowing-down process towards a successive deceleration. For this, let us briefly recapitulate Paveri-Fontana's Boltzmann-like equation (for a detailed discussion see [12,1]) and then introduce the modified interaction term.

Let $g(x,v,w,t)$ denote the one-vehicle distribution function for vehicles with desired speed w in the phase space spanned by x,v,w,t, where $g(x,v,w,t)dx\, dv\, dw$ denotes the number of vehicles at time t, in position dx around x, and actual speed dv around v with desired speed dw around w. The road is assumed to be a one-dimensional, unidirectional lane, but passing is allowed. This can be conceived as a coarse-grained multilane road where an average over the different lanes has been taken.

The *one-vehicle speed distribution function* $f(x,v,t)$ and the *one-vehicle desired speed distribution function* $f_0(x,w,t)$ are given by

$$f(x,v,t)=\int_0^{+\infty} dw\; g(x,v,w,t), \tag{2.1}$$

$$f_0(x,w,t)=\int_0^{+\infty} dv\; g(x,v,w,t). \tag{2.2}$$

The vehicular concentration $c(x,t)$, the average velocity $\overline{v(x,t)}$, the average desired velocity $\overline{w(x,t)}$, and the flow $q(x,t)$ are then defined as

$$c(x,t)=\int_0^{+\infty} dw \int_0^{+\infty} dv\; g(x,v,w,t), \tag{2.3}$$

$$\overline{v(x,t)}=\frac{\int_0^{+\infty}dw\int_0^{+\infty}dv\; vg(x,v,w,t)}{c(x,t)}, \tag{2.4}$$

$$\overline{w(x,t)}=\frac{\int_0^{+\infty}dw\int_0^{+\infty}dv\; wg(x,v,w,t)}{c(x,t)}, \tag{2.5}$$

$$q(x,t)=c(x,t)\overline{v(x,t)}. \tag{2.6}$$

Paveri-Fontana's Boltzmann-like traffic equation reads

$$\begin{aligned}\left(\frac{\partial}{\partial t}+v\,\frac{\partial}{\partial x}\right)g+\frac{\partial}{\partial v}\left(\frac{w-v}{T}\,g\right)\\ =f(x,v,t)\int_0^{+\infty}dv'(1-P)(v'-v)g(x,v',w,t)\\ -g(x,v,w,t)\int_0^{v}dv'(1-P)(v-v')f(x,v',t),\end{aligned} \tag{2.7}$$

where the following assumptions have been made.

(i) The slowing-down process has a probability $(1-P)$, where P denotes the probability of passing, $0\leqslant P\leqslant 1$. If the fast car passes the slow one, its velocity is not affected.

(ii) The velocity of the slow car is unaffected by the interaction or by the fact of being passed.

(iii) Cars are regarded as pointlike objects, so the vehicle length can be neglected.

(iv) The slowing-down process is instantaneous, i.e., there is no braking time.

(v) Only two-vehicle interactions are considered; multivehicle interactions are excluded.

(vi) One assumes "vehicular chaos", i.e., vehicles are not correlated,

$$g_2(x,v,w,x',v',w',t)\simeq g(x,v,w,t)g(x',v',w',t), \tag{2.8}$$

where g_2 denotes the two-vehicle distribution function.

The first part of the collision integral of Eq. (2.7) describes the gain of the phase-space element, i.e., vehicles with velocity $v'\geqslant v$ collide with vehicles with velocity v, while the second term describes the loss of the phase space element, i.e., vehicles with velocity v collide with vehicles with even slower velocity v'. Furthermore, it is assumed that no driver changes his desired speed, i.e.,

$$\frac{dw}{dt}=0, \tag{2.9}$$

and that the acceleration of each car is modeled by

$$\frac{dv}{dt}=\frac{w-v}{T}, \tag{2.10}$$

i.e., the drivers approach their desired speed exponentially in time, with time constant T. (T might be a function of $c,\overline{v}$, see, for example, [13].)

The probability of passing is usually chosen to be density dependent, for example, $P(c)=1-c/\hat{c}$ ($\hat{c}$ denotes the maximal density [11]), but additional velocity and variance dependences have been proposed in [13]. The assumption of an instantaneous interaction is approximately valid for processes where the slowing-down time $\Delta\tau$ and the length $v\Delta\tau$ are short compared to the characteristic time and length scales involved. Having made the assumption of vehicular chaos, the microscopic equation is valid only for dilute traffic.

In [1] we have already extended assumption (iii) to vehicles with finite length including a velocity-dependent safety distance. Hence we have introduced the required car length $d(v)=l+\tau v$, where l is the average car length, v is the velocity of the car, and τ is a reaction time. Now we want to relax assumption (iv), i.e., the assumption that the slowing-down process is instantaneous. For this, we describe the interaction process as follows [note that we take into account the required car length $d(v)$ from the very beginning]. Let $v_<$ and $v_>$ denote the actual speed of the slow car and the fast car, respectively. When the fast car reaches the slow car from behind the slow car is a distance $d(v_>)$ ahead. The two vehicles interact with a probability $1-P$ [assumption (i)]. When an interaction takes place the slow car remains unaffected, i.e., it retains its velocity $v_<$ [assumption (ii)], whereas the fast car changes its velocity to $\Phi(v_<,v_>)$, with $0\leqslant\Phi< v_>$. In the original version of the collision term the fast vehicle adopts the velocity of the preceeding slow vehicle, i.e., $\Phi(v_<,v_>)=v_<$. In order to investigate the effect of a more moderate deceleration we will choose below $\Phi(v_<,v_>)=\frac{1}{2}(v_<+v_>)$. For other possible functions Φ the calculations can be done analogously. For example, a function Φ' with $\Phi'(v_<,v_>)\leqslant v_<$ would describe an "overbraking" of the fast vehicle. The remaining assumptions (v) and (vi) are still assumed to be valid.

The modified collision integral on the right-hand side (RHS) of Eq. (2.7) is then given by

$$\begin{aligned}\left(\frac{\partial g}{\partial t}\right)_{\mathrm{coll}}=&\int\int_{0\leqslant v_1\leqslant v_3}dv_1dv_3\sigma(x+d(v_3))(v_3-v_1)\\ &\times f(x+d(v_3),v_1,t)g(x,v_3,w,t)\delta(v-\Phi(v_1,v_3))\\ &-g(x,v,w,t)\int\int_{0\leqslant v_1\leqslant v}dv_1dv_2\sigma(x+d(v))\\ &\times(v-v_1)f(x+d(v),v_1,t)\delta(v_2-\Phi(v_1,v)).\end{aligned} \tag{2.11}$$

The first part describes the gain of the phase-space element around v,w, i.e., vehicles with velocity v_3 collide with vehicles with velocity $v_1\leqslant v_3$ such that the velocity of the faster vehicle after the collision is $v=\Phi(v_1,v_3)$. The second term describes the loss of the phase-space element around v,w, i.e., vehicles with velocity v collide with vehicles with slower velocity $v_1\leqslant v$ and are scattered to velocities $v_2=\Phi(v_1,v)$.

When taking into account the finite car length, the effective volume is reduced and thus the collision frequency is increased. As in [1], we incorporate this by changing the cross section in Eq. (2.7) from $1-P$ to the modified cross section $\sigma=\chi(c,\overline{v})[1-P(c)]$ in Eq. (2.11), with

$$\chi(c,\overline{v}) = \frac{1}{1 - c d(\overline{v})} = \frac{1}{1 - c(l + \tau\overline{v})}. \tag{2.12}$$

Since the hindrance is now a distance $d(v)$ ahead of the fast car the cross section depends on the macroscopic quantities at this point. This is implied by the notation $\sigma(x+d(v))$. Later on we will Taylor expand σ around the position x and use

$$\begin{aligned}\partial_x\sigma &= (1-P)\chi^2[d(\overline{v})\partial_x c + c\tau\partial_x\overline{v}] - \chi P'\partial_x c\\ &= \eta\partial_x c + \zeta\partial_x\overline{v},\end{aligned} \tag{2.13}$$

with $\eta = [\sigma d(\overline{v}) - P']\chi$, $\zeta = \sigma\chi c\tau$, and P' denotes the derivative of P with respect to its argunent. Note that $\eta,\zeta \geqslant 0$.

Notice also that the interaction is still instantaneous, but the slowing-down process is now stepwise. When a fast car interacts with a slow car its speed is reduced to $v = \Phi(v_<, v_>)$. After the interaction the fast car is still behind the slow car, but has now the required length $d(v) \leqslant d(v_>)$. The fast car moves on before another collision can take place.

III. MACROSCOPIC EQUATIONS

As in [1], we find a hierachy of moment equations by taking the moments of Eq. (2.7), but we now have to be more careful when considering the moments of the collision integral. In order to evaluate the integrals we again neglect third- and higher-order terms in the cumulant expansion of the distribution function. This amounts to an approximation of the local equilibrium function by a normal distribution, i.e., we assume that the vehicles with desired velocity w are normally distributed and that the local equilibrium distribution in the actual velocity is given by a Gaussian (the latter is supported by experimental data [13,19–23]). One first defines

$$\delta v := v - \overline{v}, \quad \delta w := w - \overline{w} \tag{3.1}$$

and

$$\Theta_{vv} := \overline{(\delta v)^2}, \quad \Theta_{ww} := \overline{(\delta w)^2}, \quad \Theta_{vw} := \overline{\delta v\,\delta w}, \tag{3.2}$$

where the overbar denotes the normalized average with respect to the distribution function $g(v,w)$.

A. The continuity equation

Integration of the collision integral (2.11) over dv gives

$$\begin{aligned}
&\int_0^\infty dv \int\int_{0\leqslant v_1\leqslant v_3} dv_1 dv_3 \sigma(v_3 - v_1) f(x+d(v_3), v_1, t) g(x, v_3, w, t)\,\delta(v - \Phi(v_1, v_3))\\
&\quad - \int_0^\infty dv \int\int_{0\leqslant v_1\leqslant v} dv_1 dv_2 \sigma(v - v_1) f(x+d(v), v_1, t) g(x, v, w, t)\,\delta(v_2 - \Phi(v_1, v))\\
&= \int_0^\infty dv_2 \int\int_{0\leqslant v_1\leqslant v_3} dv_1 dv_3 \sigma(v_3 - v_1) f(x+d(v_3), v_1, t) g(x, v_3, w, t)\,\delta(v_2 - \Phi(v_1, v_3))\\
&\quad - \int_0^\infty dv_3 \int\int_{0\leqslant v_1\leqslant v_3} dv_1 dv_2 \sigma(v_3 - v_1) f(x+d(v), v_1, t) g(x, v, w, t)\,\delta(v_2 - \Phi(v_1, v_3))\\
&= \int\int\int_{0\leqslant v_1\leqslant v_2\leqslant v_3} dv_1 dv_2 dv_3 \sigma(v_3 - v_1) f(x+d(v_3), v_1, t) g(x, v_3, w, t)\,\delta(v_2 - \Phi(v_1, v_3))\\
&\quad - \int\int\int_{0\leqslant v_1\leqslant v_2\leqslant v_3} dv_1 dv_2 dv_3 \sigma(v_3 - v_1) f(x+d(v_3), v_1, t) g(x, v_3, w, t)\,\delta(v_2 - \Phi(v_1, v_3)) = 0.
\end{aligned} \tag{3.3}$$

The contribution of the relaxation term in Eq. (2.7) disappears due to vanishing surface terms. Thus we find

$$\frac{\partial}{\partial t} f_0(x,w,t) + \frac{\partial}{\partial x}[\overline{v(x,w,t)} f_0(x,w,t)] = 0, \tag{3.4}$$

where $\overline{v(x,w,t)}$ is defined as

$$\overline{v(x,w,t)} = \frac{\int_0^{+\infty} dv\; v g(x,v,w,t)}{f_0(x,w,t)}. \tag{3.5}$$

Equation (2.11) is a continuity equation for each desired speed w separately and has certainly also been found for the original equation (2.7); see [12,1]. A further integration over dw leads again to the continuity equation

$$\frac{\partial}{\partial t} c + \frac{\partial}{\partial x}(c\overline{v}) = 0. \tag{3.6}$$

B. The mean desired velocity equation

As in [12,1], integrating [LHS of Eq. (2.7)]=[Eq. (2.11)] over $dv\ dw\ w$ leads to

$$\frac{\partial}{\partial t}(c\overline{w})+\frac{\partial}{\partial x}(c\overline{vw})=0 \tag{3.7}$$

and finally to

$$\partial_t\overline{w}+\overline{v}\partial_x\overline{w}=-\frac{\Theta_{vw}}{c}\partial_x c-\partial_x\Theta_{vw}. \tag{3.8}$$

C. The mean actual velocity

We now take $\Phi(v_<,v_>)=\frac{1}{2}(v_<+v_>)$. For the LHS of Eq. (2.7), we again find after an integration over $dv\ dw\ v$,

$$\frac{\partial}{\partial t}(c\overline{v})+\frac{\partial}{\partial x}(c\overline{v^2})+\frac{c}{T}(\overline{v}-\overline{w}), \tag{3.9}$$

whereas for the interaction term Eq. (2.11) one gets

$$\begin{aligned}&\int_0^\infty dv\int\int_{0\le v_1\le v_3}dv_1dv_3\sigma(x+d(v_3))v(v_3-v_1)\\&\quad\times f(x+d(v_3),v_1,t)f(x,v_3,t)\,\delta\left(v-\frac{v_1+v_3}{2}\right)\\&\quad-\int_0^\infty dv\int\int_{0\le v_1\le v}dv_1dv_2\sigma(x+d(v))v(v-v_1)\\&\quad\times f(x+d(v),v_1,t)f(x,v,t)\,\delta\left(v_2-\frac{v_1+v}{2}\right)\\&\approx-\tfrac{1}{2}\sigma c^2\Theta_{vv}+\tfrac{1}{2}c(\alpha_1\partial_x c+\alpha_2\partial_x\overline{v}+\alpha_3\partial_x\Theta_{vv})\\&\quad+\tfrac{1}{2}\alpha_4\partial_x\sigma.\end{aligned} \tag{3.10}$$

In Appendix A we show the main steps of the calculation. We just remark that we have Taylor expanded the distribution function around x, keeping only the linear terms, and that we have approximated the local equilibrium solution by a normal distribution. The coefficients $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ are given below.

Equation (3.10), together with Eqs. (2.13) and (3.9), leads eventually to

$$\begin{aligned}\partial_t\overline{v}+\overline{v}\partial_x\overline{v}=&\left(\frac{1}{2}\alpha_1+\frac{1}{2}\eta\alpha_4-\frac{\Theta_{vv}}{c}\right)\partial_x c+\frac{1}{2}(\alpha_2+\zeta\alpha_4)\partial_x\overline{v}\\&+\left(\frac{1}{2}\alpha_3-1\right)\partial_x\Theta_{vv}+\frac{\overline{w}-\overline{v}}{T}-\frac{1}{2}\sigma\Theta_{vv},\end{aligned} \tag{3.11}$$

with

$$\alpha_1=-\sigma\left(l+\tau\left(\frac{2}{\sqrt{\pi}}\sqrt{\Theta_{vv}}+\overline{v}\right)\right)\Theta_{vv}. \tag{3.12}$$

$$\alpha_2=\sigma\left(\frac{2}{\sqrt{\pi}}\sqrt{\Theta_{vv}}(l+\tau\overline{v})+\tau\Theta_{vv}\right)c, \tag{3.13}$$

$$\alpha_3=-\sigma\frac{1}{2}\left(l+\tau\left(\frac{1}{\sqrt{\pi}}\sqrt{\Theta_{vv}}+\overline{v}\right)\right)c, \tag{3.14}$$

$$\alpha_4=-c\Theta_{vv}(l+\tau\overline{v})-\tau\frac{2}{\sqrt{\pi}}c\Theta_{vv}\sqrt{\Theta_{vv}}. \tag{3.15}$$

Comparing these equations with the respective mean velocity equation previously derived [1], we immediately notice that the coefficients α_i are identical, the only difference being the factor $\frac{1}{2}$ in front of the α_i's and in front of the interaction term on the far-right-hand side of Eq. (3.11).

D. The variance of the actual velocity

Integration of the LHS of Eq. (2.7) over $dv\ dw\ v^2$ yields, after using Eqs. (3.8) and (3.11),

$$\begin{aligned}&\partial_t(c\Theta_{vv})+\overline{v}\partial_x(c\Theta_{vv})+3c\Theta_{vv}\partial_x\overline{v}+\partial_x[c\overline{(\partial v)^3}]\\&+\frac{2c}{T}(\Theta_{vv}-\Theta_{vw})-\sigma c^2\overline{v}\Theta_{vv}+c\overline{v}[(\alpha_1+\eta\alpha_4)\partial_x c\\&+(\alpha_2+\zeta\alpha_4)\partial_x\overline{v}+\alpha_3\partial_x\Theta_{vv}],\end{aligned} \tag{3.16}$$

whereas for the interaction term (2.11) one gets

$$\begin{aligned}&\int_0^\infty dv\int\int_{0\le v_1\le v_3}dv_1dv_3\sigma(x+d(v_3))v^2(v_3-v_1)\\&\quad\times f(x+d(v_3),v_1,t)f(x,v_3,t)\,\delta\left(v-\frac{v_1+v_3}{2}\right)\\&\quad-\int_0^\infty dv\int\int_{0\le v_1\le v}dv_1dv_2\sigma(x+d(v))v^2(v-v_1)\\&\quad\times f(x+d(v),v_1,t)f(x,v,t)\,\delta\left(v_2-\frac{v_1+v}{2}\right)\\&\approx-\sigma\left(c^2\overline{v}\Theta_{vv}+\frac{1}{\sqrt{\pi}}c^2\Theta_{vv}\sqrt{\Theta_{vv}}\right)\\&\quad+\xi_1\partial_x c+\xi_2\partial_x\overline{v}+\xi_3\partial_x\Theta_{vv}+\xi_4\partial_x\sigma,\end{aligned} \tag{3.17}$$

where $\xi_1,\xi_2,\xi_3\xi_4$ have a similar structure to the α,'s. The main steps of the calculation are given in Appendix B. To find the coefficients ξ_i we have again used a cumulant expansion and have discarded third- and higher-order terms. Together with Eq. (3.16) (discarding the third-order cumulant) we get

$$\begin{aligned}\partial_t\Theta_{vv}+\overline{v}\partial_x\Theta_{vv}=&\frac{2}{T}(\Theta_{vw}-\Theta_{vv})-\frac{1}{\sqrt{\pi}}\sigma c\Theta_{vv}\sqrt{\Theta_{vv}}\\&+(\tilde{\beta}_1+\eta\tilde{\beta}_4)\partial_x c+(\tilde{\beta}_2+\zeta\tilde{\beta}_4-2\Theta_{vv})\partial_x\overline{v}\\&+\tilde{\beta}_3\partial_x\Theta_{vv},\end{aligned} \tag{3.18}$$

with

$$\tilde{\beta}_1=-\sigma\left(\frac{1}{\sqrt{\pi}}\Theta_{vv}\sqrt{\Theta_{vv}}(l+\tau\overline{v})+\frac{5}{4}\Theta_{vv}^2\tau\right), \tag{3.19}$$

$$\tilde{\beta}_2=\sigma\left(\frac{1}{4}\Theta_{vv}(l+\tau\overline{v})+\frac{3}{2\sqrt{\pi}}\Theta_{vv}\sqrt{\Theta_{vv}}\tau\right)c, \tag{3.20}$$

$$\widetilde{\beta}_3=\sigma\left(\frac{1}{4\sqrt{\pi}}\sqrt{\Theta_{vv}}(l+\tau\overline{v})-\frac{1}{8}\Theta_{vv}\tau\right)c, \tag{3.21}$$

$$\widetilde{\beta}_4=-c\Theta_{vv}\left(\frac{1}{\sqrt{\pi}}\sqrt{\Theta_{vv}}(l+\tau\overline{v})+\frac{5}{4}\Theta_{vv}\tau\right). \tag{3.22}$$

Now the coefficients $\widetilde{\beta}_i$ have slightly changed in comparison to the coefficients β_i derived in [1]. More important is the appearance of the second highly nonlinear term on the RHS of Eq. (3.18). The negative sign of this term yields an impeding effect on the dynamical behavior of the variance, similar to the last term of the covariance equation (3.27) (see below).

E. The variance of the desired velocity

As in [12,1], an integration of [LHS of Eq. (2.7)]=[Eq. (2.11)] over $dv\ dw\ w^2$ leads to

$$\partial_t(c\Theta_{ww})+\overline{v}\partial_x(c\Theta_{ww})+2c\Theta_{vw}\partial_x\overline{w}+c\Theta_{ww}\partial_x\overline{v}+\partial_x[c\overline{(\delta w)^2\delta v}]=0 \tag{3.23}$$

and after discarding third-order cumulant to

$$\partial_t\Theta_{ww}+\overline{v}\partial_x\Theta_{ww}+2\Theta_{vw}\partial_x\overline{w}=0. \tag{3.24}$$

F. The covariance equation

For the LHS of Eq. (2.7), we again find after an integration over $dv\ dw\ vw$ and using Eqs. (3.8) and (3.11)

$$\begin{aligned}&\partial_t(c\Theta_{vw})+\overline{v}\partial_x(c\Theta_{vw})+2c\Theta_{vw}\partial_x\overline{v}+c\Theta_{vv}\partial_x\overline{w}\\&+\partial_x(c\overline{(\delta v)^2\delta w})-\frac{1}{2}\sigma c^2\overline{w}\Theta_{vv}+\frac{c}{T}(\Theta_{vw}-\Theta_{ww})\\&+\frac{1}{2}c\overline{w}[(\alpha_1+\eta\alpha_4)\partial_x c+(\alpha_2+\zeta\alpha_4)\partial_x\overline{v}+\alpha_3\partial_x\Theta_{vv}].\end{aligned} \tag{3.25}$$

For the interaction term (2.11) one gets

$$\begin{aligned}&\int_0^\infty du\ w\int_0^\infty dv\int\int_{0\leq v_1\leq v_3}dv_1dv_3\sigma(x+d(v_3))v(v_3-v_1)f(x+d(v_3),v_1,t)g(x,v_3,w,t)\,\delta\left(v-\frac{v_1+v_3}{2}\right)\\&-\int_0^\infty dw\ w\int_0^\infty dv\int\int_{0\leq v_1\leq v}dv_1dv_2\sigma(x+d(v))v(v-v_1)f(x+d(v),v_1,t)g(x,v,w,t)\,\delta\left(v_2-\frac{v_1+v}{2}\right)\\&\approx\frac{1}{2}\int_0^\infty dw\ w\int_0^\infty dv_3\int_0^{v_3}dv_1(\sigma+d(v_3)\partial_x\sigma)(v_1-v_3)(v_3-v_1)[f(x,v_1,t)+d(v_3)\partial_x f(x,v_1,t)]g(x,v_3,w,t)\\&\approx\frac{1}{2}\sigma\left(c^2\overline{w}\Theta_{vv}+\frac{2}{\sqrt{\pi}}c^2\Theta_{vw}\sqrt{\Theta_{vv}}\right)+\frac{1}{2}(\vartheta_1\partial_x c+\vartheta_2\partial_x\overline{v}+\vartheta_3\partial_x\Theta_{vv}+\vartheta_4\partial_x\sigma),\end{aligned} \tag{3.26}$$

where $\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4$, have a similar structure to the α's. Together with Eq. (3.25) (discarding the third-order cumulant) we get

$$\begin{aligned}\partial_t\Theta_{vw}+\overline{v}\partial_x\Theta_{vw}=&\frac{1}{2}(\gamma_1+\eta\gamma_4)\partial_x c\\&+\left(\frac{1}{2}\gamma_2+\frac{1}{2}\zeta\gamma_4-\Theta_{vw}\right)\partial_x\overline{v}-\Theta_{vv}\partial_x\overline{w}\\&+\frac{1}{2}\gamma_3\partial_x\Theta_{vv}+\frac{1}{T}(\Theta_{ww}-\Theta_{vw})\\&-\frac{1}{2}\sigma\frac{2}{\sqrt{\pi}}c\Theta_{vw}\sqrt{\Theta_{vv}},\end{aligned} \tag{3.27}$$

with

$$\gamma_1=-\sigma\left(\frac{2}{\sqrt{\pi}}\Theta_{vw}\sqrt{\Theta_{vv}}(l+\tau\overline{v})+\tau\Theta_{vw}\Theta_{vv}\right), \tag{3.28}$$

$$\gamma_2=\sigma\left((l+\tau\overline{v})\Theta_{vw}+\tau\frac{3}{\sqrt{\pi}}\Theta_{vw}\sqrt{\Theta_{vv}}\right)c, \tag{3.29}$$

$$\begin{aligned}&\gamma_3=-\sigma\left(\frac{1}{2\sqrt{\pi}}\frac{\Theta_{vw}}{\sqrt{\Theta_{vv}}}(l+\tau\overline{v})+\frac{1}{2}\tau\Theta_{vw}\right)c,\\&\gamma_4=-\frac{2}{\sqrt{\pi}}c\Theta_{vw}\sqrt{\Theta_{vv}}(l+\tau\overline{v})-2\tau c\Theta_{vv}\Theta_{vw}.\end{aligned} \tag{3.30}$$

Comparing Eq. (3.27) with the former version in [1], we find that again the coefficients γ_i are identical and that just the factor $\frac{1}{2}$ has emerged in front of the γ's and the far-right-hand term of Eq. (3.27).

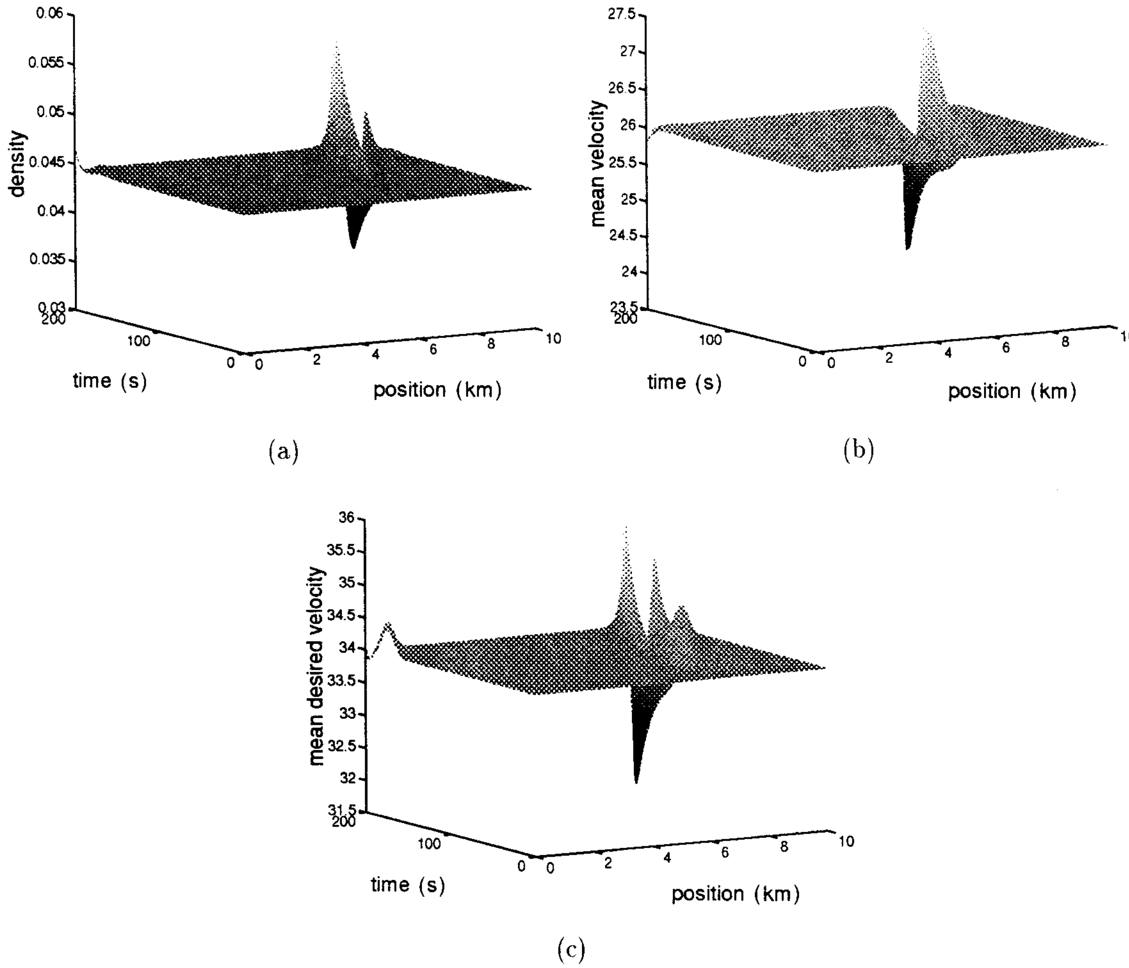

FIG. 1. Time evolution of the traffic flow starting from a homogeneous state with a small perturbation of the desired velocity variance using Eqs. (3.6), (3.8), (3.11), (3.18), (3.24), and (3.27): (a) density ρ, (b) mean velocity $\overline{v}$, and (c) mean desired velocity $\overline{w}$.

G. The homogeneous solution

The homogeneous solution for the system of partial differential equations (3.6), (3.8), (3.11), (3.18), (3.24), and (3.27) is found to be

$$\overline{w}-\overline{v}=\frac{T}{2}\sigma c\Theta_{vv}, \tag{3.31}$$

$$\Theta_{vw}=\Theta_{vv}\left(1+\frac{T}{2}\frac{\sigma}{\sqrt{\pi}}c\sqrt{\Theta_{vv}}\right), \tag{3.32}$$

$$\Theta_{ww}=\Theta_{vw}\left(1+T\sigma\frac{1}{\sqrt{\pi}}c\sqrt{\Theta_{vv}}\right). \tag{3.33}$$

Thus, given certain values for c, v, and Θ_{vv}, the mean desired velocity $\overline{w}$, the variance Θ_{ww}, and the covariance Θ_{vw} are determined.

Let us now summarize the comparison. The continuity equation (3.6), the equation for the mean desired velocity (3.8), and the equation for the variance of the desired velocity (3.24) remain unchanged. In the mean velocity equation (3.11) and the covariance equation (3.27) we find a factor $\frac{1}{2}$ for the term originating from the collision integral. In Eq. (3.18) for the variance of the actual velocity the coefficients $\widetilde{\beta}_i$ have changed, but the more important feature is the term $-(1/\sqrt{\pi})\sigma c\Theta_{vv}\sqrt{\Theta_{vv}}$, which has no corresponding part in the original model (see [1]). The homogeneous solution has also changed. In Eq. (3.31) the factor $\frac{1}{2}$ appears on the RHS and Θ_{vw} and Θ_{vv} are no longer equal [Eq. (3.32)] as in the original model. Finally, in the relation between Θ_{ww} and Θ_{vw} a factor 2 is missing in front of the scattering probability σ. The numerical simulations below show that these changes and especially the different nonlinear term result in qualitatively different behavior of the variance of the actual velocity.

IV. NUMERICAL SIMULATIONS

In Figs. 1 and 2 we present numerical simulations of Eqs. (3.6), (3.8), (3.11), (3.18), (3.24) and (3.27) obtained by stepwise integration. Periodic boundary conditions are assumed and $T=300$, τ=0.5 s, and $l=5$ m. As the homogeneous solution we choose $v=26$ m/s, $c=0.8\hat{c}(\hat{v})$, and $\Theta_{vv}=10.8$ m^2/s^2. Equations (3.31)–(3.33) then yield $\overline{w}=34m/s$, $\Theta_{ww}=96.0$ m^2/s^2, and $\Theta_{vw}=25.6$ m^2/s^2. At time $t=0$ and position $x=5$ km we have added a small Gaussian-shaped perturbation to the otherwise constant desired velocity variance Θ_{ww}.

First, we observe that similar to our former model [1], a

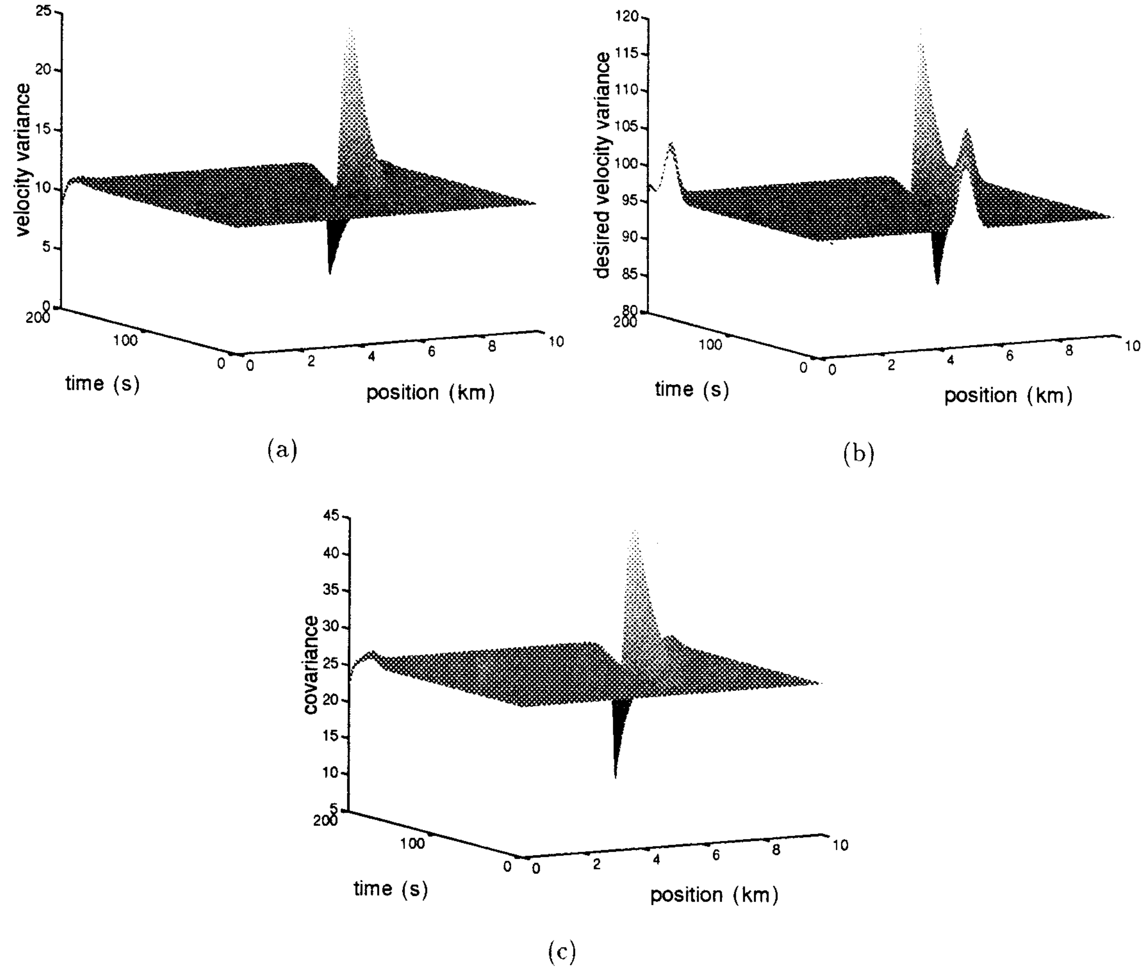

FIG. 2. Time evolution of the variances Θ_{vv}, Θ_{ww} and of the covariance Θ_{vw}: (a) variance $\overline{(\delta v)^2}$, (b) variance $\overline{(\delta w)^2}$, and (c) covariance $\overline{\delta v\, \delta w}$.

traffic cluster is built, i.e., we find a region of lower density followed by a region of higher density. But now the increase of the density [Fig. 1(a)] and the decrease of the mean velocity [Fig. 1(b)] are much more moderate, as we expect from the modified collision integral. The other dynamical quantities [Figs. 1(c), and 2(a)–2c] behave in a similar moderate way. The striking different feature is now the decrease of the velocity variance in the jam region [Fig. 2(a)], in contrast to the increasing behavior of the velocity variance in the former model. This is in agreement with empirical data investigated by Helbing [2]. He finds that, although there is a small peak in the velocity variance at the onset of a traffic jam due to larger fluctuations, the velocity variance decreases in the high-density jam region. In the former model we assumed a sharp change in the velocity when a fast car reaches a slow car. Now vehicles reaching the rear end of the high-density region adapt their velocity successively to the lower mean velocity, thus the velocity variance is lowered. In Fig. 3 we have plotted the rescaled density c, the rescaled mean velocity $\overline{v}$, and the rescaled velocity variance Θ_{vv} at time $t=200$ s (periodic boundary conditions). In Fig. 2(c) we observe that the covariance is lower in the jam region, i.e., more drivers drive with a lower velocity than their desired velocity, while the covariance is higher in the preceding low-density region, meaning that it is now easier to reach one's desired velocity. In order to interpret the dynamical behavior of the mean desired velocity and the desired velocity variance we show these rescaled quantities together with the rescaled density at time $t=200$ in Fig. 4. A high desired velocity variance represents a mixing of desired velocity ''classes,'' while a low desired velocity variance means that only drivers with similar desired velocities are present. At the rear end of the density cluster we find a higher mean desired velocity and a lower desired velocity variance, i.e., more drivers with a high desired velocity reach the rear end per time unit than drivers with a low desired velocity. At the front end of the jam we observe a low mean desired velocity and a high variance. At this point the cars reach the free driving region, i.e., we find all kinds of driver characters, but the more timid drivers stay behind. The more aggressive drivers accumulate again at the front end of the low-density region. To summarize the simulation results, we observe that the overall behavior of the dynamical quantities is similar to the former model, except that the velocity variance shows now a more realistic behavior.

V. CONCLUSION

In order to model a more realistic slowing-down process we have presented a modified Boltzmann-like traffic equa-

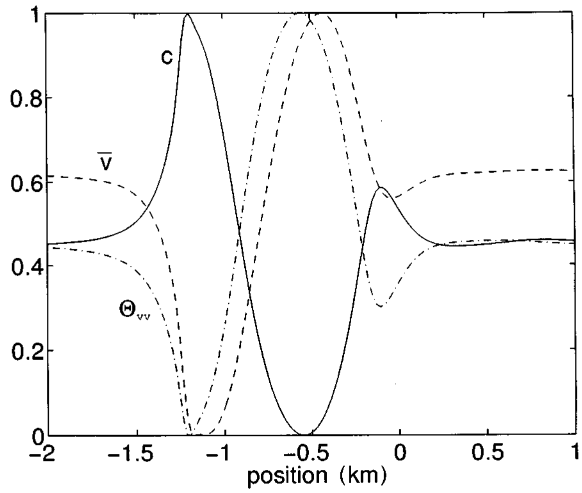

FIG. 3. Rescaled dynamical quatities at $t=200$ s (note the periodic boundary conditions): density c (solid line), mean velocity $\overline{v}$ (dashed line), and velocity variance Θ_{vv} (dash-dotted line).

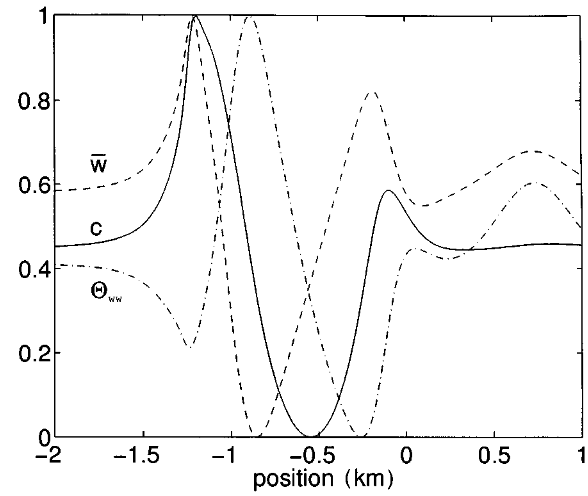

FIG. 4. Rescaled dynamical quatities at $t=200$ s (note the periodic boundary conditions): density c (solid line,) mean desired velocity $\overline{w}$ (dashed line), and desired velocity variance Θ_{ww} (dash-dotted line).

tion. In contrast to Paveri-Fontana's [12] original version, the deceleration is now a successive process. We have then derived a macroscopic traffic model and have found that the most important different feature is a nonlinear term in the velocity variance equation. Numerical simulations show that again a traffic cluster is built, albeit the dynamical evolution is now more moderate. More striking is the dynamical behavior of the velocity variance, which is now complementary to our former model. Whereas in the former model the velocity variance possesses a sharp peak in the high-density region, we now find a smaller velocity variance in the cluster. This behavior is qualitatively supported by empirical traffic data [2] and can be explained by the interaction process since vehicles now adjust their velocity stepwise to the velocity of the leading vehicles. In order to obtain macroscopic quantities from measurements one has to average over a certain time inteval. Instantaneous slowing down means that the time of deceleration is small compared to the averaging time. When fast vehicles reach slower vehicles they have higher velocities before the slowing-down process and lower velocities after, i.e., the macroscopic velocity variance is higher. But since drivers also react on traffic conditions far ahead, the slowing-down time can exceed the averaging time, i.e., the deceleration can no longer be regarded as being instantaneous. The two models represent two extreme ways of modeling the slowing-down processes: One is rather sudden, whereas the other is highly ordered and moderate. Up to now, we have not taken into account the case of panic braking, i.e., a fast vehicle finds a slower leading vehicle within its safety distance, or the case of imperfect braking, i.e., vehicles slow down to velocities slower than the velocity of their leading vehicle. The former has been worked out by Nelson [10] by generalizing the vehicular chaos assumption, whereas the latter has been treated by Helbing [2]. Both cases result in sudden velocity changes yielding a higher variance and might therefore cancel to some extent the effects of the successive slowing-down process discussed here, but the overall macroscopic behavior is apparently governed by a more moderate velocity adjustment.

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APPENDIX A: THE MEAN ACTUAL VELOCITY

Here we just sketch some of the steps to get from the first line of Eq. (3.10) to the second line:

$$\int_0^\infty dv \int\int_{0\le v_1\le v_3} dv_1 dv_3 \sigma(x+d(v_3)) v(v_3-v_1) f(x+d(v_3),v_1,t) f(x,v_3,t)\, \delta\!\left(v-\frac{v_1+v_3}{2}\right)$$
$$-\int_0^\infty dv \int\int_{0\le v_1\le v} dv_1 dv_2 \sigma(x+d(v)) v(v-v_1) f(x+d(v),v_1,t) f(x,v,t)\, \delta\!\left(v_2-\frac{v_1+v}{2}\right)$$

$$=\int_0^\infty dv_2 \int\int_{0\le v_1\le v_2\le v_3} dv_1 dv_3 \sigma(x+d(v_3)) v_2(v_3-v_1) f(x+d(v_3),v_1,t) f(x,v_3,t)\, \delta\!\left(v_2-\frac{v_1+v_3}{2}\right)$$
$$-\int_0^\infty dv_2 \int\int_{0\le v_1\le v_2\le v_3} dv_1 dv_3 \sigma(x+d(v_3)) v_3(v_3-v_1) f(x+d(v_3),v_1,t) f(x,v_3,t)\, \delta\!\left(v_2-\frac{v_1+v_3}{2}\right)$$

$$\approx \tfrac{1}{2}\int_0^\infty dv_3\int_0^{v_3} dv_1\sigma(x+d(v_3))(v_1-v_3)(v_3-v_1)[f(x,v_1,t)+d(v_3)\partial_x f(x,v_1,t)]f(x,v_3,t)$$

$$\approx \tfrac{1}{2}\sigma\int_0^\infty dv_3\int_0^\infty dv_1(v_1v_3-v_1^2)f(x,v_1,t)f(x,v_3,t)-\tfrac{1}{2}\partial_x\sigma\int_0^\infty dv_3\int_0^{v_3} dv_1 d(v_3)(v_3-v_1)^2 f(x,v_1,t)f(x,v_3,t)$$

$$+\tfrac{1}{2}\sigma\int_0^\infty dv_3\int_0^{v_3} dv_1(2v_1v_3-v_1^2-v_3)d(v_3)f(x,v_3,t)\partial_x f(x,v_1,t)$$

$$\approx -\tfrac{1}{2}\sigma c^2\Theta_{vv}+\tfrac{1}{2}c(\alpha_1\partial_x c+\alpha_2\partial_x\overline{v+}\alpha_3\partial_x\Theta_{vv})+\tfrac{1}{2}\alpha_4\partial x\sigma. \tag{A1}$$

In going from the second to the third line we have Taylor expanded the distribution function around x and only kept the linear terms. Between line 3 and line 4 we have Taylor expanded the cross section σ. For the detailed calculations of the integrals in line 4 (and similar integrals below), we refer to [1], where similar expressions have been evaluated for the original collision integral. In order to derive these corrections we have approximated the local equilibrium solution by a normal distribution.

APPENDIX B: THE VARIANCE OF THE ACTUAL VELOCITY

To get to the last line of Eq. (3.17) we make the following steps:

$$\int_0^\infty dv\int\int_{0\leqslant v_1\leqslant v_3} dv_1dv_3\sigma(x+d(v_3))v^2(v_3-v_1)f(x+d(v_3),v_1,t)f(x,v_3,t)\,\delta\!\left(v-\frac{v_1+v_3}{2}\right)$$

$$-\int_0^\infty dv\int\int_{0\leqslant v_1\leqslant v} dv_1dv_2\sigma(x+d(v))v^2(v-v_1)f(x+d(v),v_1,t)f(x,v,t)\,\delta\!\left(v_2-\frac{v_1+v}{2}\right)$$

$$=\int_0^\infty dv_2\int\int_{0\leqslant v_1\leqslant v_2\leqslant v_3} dv_1dv_3\sigma(x+d(v_3))v_2^2(v_3-v_1)f(x+d(v_3),v_1,t)f(x,v_3,t)\,\delta\!\left(v_2-\frac{v_1+v_3}{2}\right)$$

$$-\int_0^\infty dv_2\int\int_{0\leqslant v_1\leqslant v_2\leqslant v_3} dv_1dv_3\sigma(x+d(v_3))v_3^2(v_3-v_1)f(x+d(v_3),v_1,t)f(x,v_3,t)\,\delta\!\left(v_2-\frac{v_1+v_3}{2}\right)$$

$$\approx\tfrac{1}{4}\int_0^\infty dv_3\int_0^{v_3} dv_1\sigma(x+d(v_3))v_1^2(v_3-v_1)[f(x,v_1,t)+d(v_3)\partial_x f(x,v_1,t)]f(x,v_3,t)$$

$$-\tfrac{3}{4}\int_0^\infty dv_3\int_0^{v_3} dv_1\sigma(x+d(v_3))v_3^2(v_3-v_1)[f(x,v_1,t)+d(v_3)\partial_x f(x,v_1,t)]f(x,v_3,t)$$

$$+\tfrac{1}{2}\int_0^\infty dv_3\int_0^{v_3} dv_1\sigma(x+d(v_3))v_1v_3(v_3-v_1)[f(x,v_1,t)+d(v_3)\partial_x f(x,v_1,t)]f(x,v_3,t)$$

$$\approx\tfrac{1}{4}\int_0^\infty dv_3\int_0^\infty dv_1(\sigma+d(v_3)\partial_x\sigma)v_1^2(v_3-v_1)f(x,v_1,t)f(x,v_3,t)$$

$$+\tfrac{1}{2}\int_0^\infty dv_3\int_0^{v_3} dv_1(\sigma+d(v_3)\partial_x\sigma)(2v_3^2v_1-v_3v_1^2-v_3^3)f(x,v_1,t)f(x,v_3,t)$$

$$+\int_0^\infty dv_3\int_0^{v_3} dv_1(\sigma+d(v_3)\partial_x\sigma)(-\tfrac{1}{4}v_3v_1^2+\tfrac{5}{4}v_3^2v_1-\tfrac{1}{4}v_1^3-\tfrac{3}{4}v_3^3)d(v_3)f(x,v_3,t)\partial_x f(x,v_1,t)$$

$$\approx-\sigma\left(c^2\overline{v}\Theta_{vv}+\frac{1}{\sqrt{\pi}}c^2\Theta_{vv}\sqrt{\Theta_{vv}}\right)+\xi_1\partial_x c+\xi_2\partial_x\overline{v+}\xi_3\partial_x\Theta_{vv}+\xi_4\partial_x\sigma. \tag{B1}$$

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